Optimal motion planning for differentially flat systems with guaranteed constraint satisfaction

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Abstract— **This research deals with the computation of optimal trajectories considering state and input constraints for linear and nonlinear systems that admit a polynomial representation through differential flatness. Based on a polynomial spline parameterization of the flat output an optimization problem in terms of the B-spline coefficients is derived that guarantees constraint satisfaction over the entire time horizon whereas classical approaches in the literature only impose the constraints on a finite time grid. As the proposed constraints are only sufficient conditions, a novel method is presented that effectively reduces their conservatism. Two numerical examples, a linear benchmark tracking problem and an optimal quadrotor maneuver, illustrate the efficiency and practicality of the presented method.**

I. Introduction

The determination of an open-loop control law that optimally steers a system from an initial state to a terminal state is a classical problem in control. When solving such optimal control problems differential flatness is a popular concept as it avoids the integration of differential equations and the problem boils down to determining the best flat output satisfying the boundary conditions and the state and input constraints. Classical approaches in the literature using differential flatness only impose these constraints on a finite time grid [1], [2]. As a result the constraints are not guaranteed to be satisfied in between the samples such that post-analysis is required. For linear systems, several methods have been proposed to guarantee constraint satisfaction for all time instances. Henrion [3] proposes a polynomial basis for the flat output and views the constrained motion planning problem as a polynomial nonnegativity problem and a sum-of-squares decomposition is sought for using semidefinite programming. Due to the limited degrees of freedom of a polynomial the solution space is limited. Moreover, a high polynomial degree may lead to unwanted oscillations in the solutions and numerical issues in solving the problem. To overcome these issues, Louembet et. al. [4] generalize this approach and use piecewise polynomial functions such that a lower polynomial degree can be used. The authors of [5] also adopt a piecewise polynomial parameterization, but in contrast to the sum-of-squares procedure of [4], they express the semi-infinite constraints by applying basis function segmentation and using the convex hull property of B-splines, leading to linear constraints. Such an approach yields only sufficient conditions and hence introduces conservatism, which can be quite severe [6].

The approaches [4] and [5] have also been applied to nonlinear systems. Louembet et. al. [4] require a polytopic inner approximation of the feasible set. Inevitably, this method introduces conservatism in the problem. Moreover, some feasible sets do not admit such a polytopic approximation, e.g. the relative complement of $\mathbb{R}^n$ in a closed convex set. For nonlinear systems that admit a polynomial representation by differential flatness, Suryawan et. al. [5] proposes a strategy to impose the semi-infinite constraints by relying on the convex hull property of splines and only keeping the linear and cubic monomial terms in the polynomial expansion. This way the optimization problem amounts to a simple QP. It should be noted, however, that this approach results in overly conservative constraints. Up to now, no general method exists for guaranteed constraint satisfaction that can cope with nonlinear systems.

This paper aims to develop an optimization approach with guaranteed constraint satisfaction over the entire time horizon for both systems that admit a polynomial representation by differential flatness, comprising all linear controllable systems and many nonlinear systems as well. Our method is based on the convex hull property of B-splines but does not rely on basis function segmentation as in [5]. Moreover, we propose a method to control the conservatism that is introduced.

Section II introduces the optimal control problem and the difficulties related to solving it. Subsequently, sections III and IV review differential flatness and B-splines as a means to overcome these issues. In Section V we cover two examples: one considers a linear benchmark from [7] and subsequently treated in [5], the second example studies minimal thrust trajectories for a quadrotor model in the vertical plane from [8].

II. General problem formulation

Consider a system governed by the differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \tag{1}$$

with states $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ and inputs $\mathbf{u}(t) \in \mathbb{R}^{n_u}$. We are interested in finding the control law $\mathbf{u}(t), t \in [0, t_\mathrm{f}]$ that steers the system from an initial state $\mathbf{x}_0$ to a terminal state $\mathbf{x}_{t_\mathrm{f}}$

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and that minimizes a performance criterion $g(\mathbf{x}, \mathbf{u}, t_\mathrm{f})$. At the same time the control law must obey the state and input constraints

$$\mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)) \geq 0, \forall t \in [0, t_\mathrm{f}]. \tag{2}$$

In this paper, we assume $\mathbf{h}$ to be polynomial in $\mathbf{x}$ and $\mathbf{u}$.

The difficulty of solving the stated optimal control problem is twofold. Firstly, the differential equation (1) requires integration, which is costly and can be numerically quite challenging depending on the system. Secondly, the semi-infinite constraint (2) is not straightforward to impose. Usually, one resorts to imposing the constraint on a finite number of time samples, but in between the samples the constraint is not guaranteed to hold. Therefore, the solution requires post-analysis. When the constraints are violated the problem should be solved anew on a finer time grid.

For differentially flat systems, the first issue is easily overcome. For such systems, we can find an algebraic relationship between the so-called flat output and the states and inputs of the system such that numerical integration of the system dynamics is avoided (see Section III). By parameterizing the flat output of the system and using the convex hull property of B-splines we can impose the semi-infinite constraint (2). As a downside, this method inevitably injects conservatism into the problem. In Section IV we show how this conservatism can be reduced.

III. Differential flatness

The concept of differentially flat systems was initially introduced by Fliess in [9] and can be defined as follows: The system (1) is differentially flat if there exists a set of variables (the flat outputs), $\mathbf{y} \in \mathbb{R}^{n_u}$ of the form

$$\mathbf{y} = \boldsymbol{\phi}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(q)}\right)$$

such that

$$\mathbf{x} = \boldsymbol{\psi}_x\left(\mathbf{y}, \mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(r-1)}\right)$$

and

$$\mathbf{u} = \boldsymbol{\psi}_u\left(\mathbf{y}, \mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(r)}\right)$$

for some positive integers q, r.

When the system (1) is linear, then it is differentially flat if it is controllable. However, for nonlinear systems there is no systematic way to determine whether a system is differentially flat. For a catalog of flat systems from various disciplines, the reader is referred to [10]. Note that many mechatronic systems are differentially flat and often admit a polynomial representation.

By exploiting differential flatness of the system we express the problem from Section II in terms of the flat output by substituting the states and inputs with the maps $\boldsymbol{\psi}_x, \boldsymbol{\psi}_u$. We arrive at following semi-infinite optimization problem

$$\begin{aligned}
\underset{\mathbf{y}(\cdot)}{\text{minimize}} \quad & g(\boldsymbol{\psi}_x(\mathbf{y}, \ldots, \mathbf{y}^{(r-1)}), \boldsymbol{\psi}_u(\mathbf{y}, \ldots, \mathbf{y}^{(r)}), t_\mathrm{f}) \\
\text{subject to} \quad & \mathbf{y}^{(j)}(0) = \mathbf{y}_{0,j},\ j = 0, \ldots, r-1 \\
& \mathbf{y}^{(j)}(t_\mathrm{f}) = \mathbf{y}_{t_\mathrm{f},j},\ j = 0, \ldots, r-1 \\
& \mathbf{h}(\boldsymbol{\psi}_x(\mathbf{y}, \ldots, \mathbf{y}^{(r-1)}), \boldsymbol{\psi}_u(\mathbf{y}, \ldots, \mathbf{y}^{(r)})) \geq 0, \\
& \forall t \in [0, t_\mathrm{f}],
\end{aligned} \tag{3}$$

where the boundary conditions for the flat output, $\mathbf{y}_0$ and $\mathbf{y}_{t_\mathrm{f}}$, are easily determined from $\mathbf{x}_0$ and $\mathbf{x}_{t_\mathrm{f}}$.

Clearly, the problem above no longer requires an integration of the system dynamics (1). However, it still contains semi-infinite constraints. In the following section we tackle this constraint by means of a spline parameterization of the flat output. For the remainder, assume the maps $\boldsymbol{\psi}_x, \boldsymbol{\psi}_u$ are polynomial in $\mathbf{y}$, which may seem restrictive but the assumption can be relaxed as shown in the examples.

IV. B-splines

A. B-spline parameterization of the flat output

Let $\boldsymbol{\kappa} = (\kappa_0, \ldots, \kappa_{m+1})$ be a strictly increasing vector of points, k be a positive integer, and $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_m)$ be a vector of integers with $0 \leq \nu_i \leq k-1$. Then, s is a polynomial spline of order k with break points $\boldsymbol{\kappa}$ and continuity conditions $\boldsymbol{\nu}$ if there exist polynomials $p_0, \ldots, p_l$ of order k such that

$$\begin{aligned}
s(t) &= p_i(t), \quad \text{for } \kappa_i \leq t < \kappa_{i+1},\ i = 0, 1, \ldots, m-1 \\
s(t) &= p_m(t), \quad \text{for } \kappa_m \leq t \leq \kappa_{m+1},
\end{aligned}$$

and

$$p_{i-1}^{(j-1)}(\kappa_i) = p_i^{(j-1)}(\kappa_i) \quad \text{for } j = 1, \ldots, \nu_i,\ i = 1, \ldots, m\,.$$

The vector space of polynomial splines with given k, $\boldsymbol{\kappa}$ and $\boldsymbol{\nu}$ is denoted by $\Pi_{k,\boldsymbol{\kappa},\boldsymbol{\nu}}$ and has dimension $n = (m+1)k - \sum_{i=1}^{m} \nu_i$. The normalized B-spline basis of order k, defined over the knot vector

$$\mathbf{t} = (\underbrace{\kappa_0, \ldots, \kappa_0}_{k}, \underbrace{\kappa_1, \ldots, \kappa_1}_{k-\nu_1}, \ldots, \underbrace{\kappa_m, \ldots, \kappa_m}_{k-\nu_m}, \underbrace{\kappa_{m+1}, \ldots, \kappa_{m+1}}_{k})$$

is commonly used as a basis for this vector space as it has various useful properties: the basis functions are nonnegative, sum up to one (partition of unity) and have local (minimal) support [11]. It yields a stable evaluation of the functions and its derivatives. A spline $s \in \Pi_{k,\boldsymbol{\kappa},\boldsymbol{\nu}}$ with B-spline basis $\mathbf{b}_s = (b_1, \ldots, b_n)$ and (B-spline) coefficients $\mathsf{s} = (\mathsf{s}_1, \ldots, \mathsf{s}_n)$ is represented as

$$s(t) = \sum_{i}^{n} \mathsf{s}_i b_i(t).$$

The control polygon of the spline is the broken line with

$$c_i = (t_i^*, \mathsf{s}_i), i = 1, \ldots, n$$

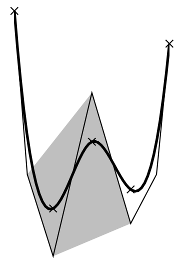

Fig. 1. A fully continuous fourth order spline (thick) with five breaks indicated by the crosses. The spline's control polygon is the broken thin line. The gray area illustrates the convex hull property for points between the second and third break.

as vertex sequence, where

$$t_i^* = \frac{t_{i+1} + \ldots + t_{i+k-1}}{k-1}, \forall i.$$

Fig. 1 illustrates a fourth order spline spline and its control polygon. The control polygon can be viewed as an exaggerated version of the spline itself [11].

With these definitions in place, we state some useful properties of polynomial splines with respect to the paper at hand.

Property 1 (Convex hull): Let s be a polynomial spline of order k with knot vector $\mathbf{t}$. From the nonnegativity, partition of unity and local support property of the B-spline basis it follows immediately that the segment $s(t), t \in [t_i, t_{i+1}]$ lies within the convex hull of its control points $c_{i-k+1}, \ldots, c_i$.

The convex hull property is also illustrated in Fig. 1. As a consequence of Property 1, a semi-infinite constraint on a polynomial spline can be imposed by only constraining its B-spline coefficients. We will rely strongly on this fact in the further course of the paper.

Property 2 (Summation): Let $q \in \Pi_{k,\boldsymbol{\kappa},\boldsymbol{\mu}}$ and $r \in \Pi_{l,\boldsymbol{\lambda},\boldsymbol{\nu}}$. Then $s = q + r \in \Pi_{\max(k,l),\boldsymbol{\xi},\boldsymbol{\omega}}$, where $\boldsymbol{\xi} = \boldsymbol{\kappa} \underset{<}{\cup} \boldsymbol{\lambda}$ the sorted, strictly increasing union of $\boldsymbol{\kappa}$ and $\boldsymbol{\lambda}$, and

$$\omega_i = \begin{cases} \min(\mu_m, \nu_n) & \text{if } \xi_i = \kappa_m = \nu_n \text{ for some } m, n \\ \mu_m & \text{if } \xi_i = \kappa_m \text{ for some } m \\ \nu_n & \text{if } \xi_i = \lambda_n \text{ for some } n. \end{cases}$$

The spline coefficients, s, of s are determined through a linear transformation of q and r:

$$\mathsf{s} = T_s^q \mathsf{q} + T_s^r \mathsf{r},$$

where T_s^q denotes the linear mapping from the B-spline basis $\mathbf{b}_q$ to $\mathbf{b}_s$

$$\mathbf{b}_q = (T_s^q)^\intercal \, \mathbf{b}_s,$$

and similarly for T_s^r.

Property 3 (Multiplication): Let $q \in \Pi_{k,\boldsymbol{\kappa},\boldsymbol{\mu}}$ and $r \in \Pi_{l,\boldsymbol{\lambda},\boldsymbol{\nu}}$. Then $s = qr \in \Pi_{k+l,\boldsymbol{\xi},\boldsymbol{\omega}}$, where $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$ are determined as in Property 2.[1] The spline coefficients, s, of s are determined through:

$$\mathsf{s} = T_s^{q \otimes r} \left(\mathsf{q} \otimes \mathsf{r} \right),$$

where $\otimes$ denotes the Kronecker product and $T_s^{q\otimes r}$ is the linear mapping from $\mathbf{b}_q \otimes \mathbf{b}_r$ to $\mathbf{b}_s$:

$$\mathbf{b}_q \otimes \mathbf{b}_r = \left(T_s^{q\otimes r}\right)^\intercal \mathbf{b}_s.$$

Properties 2 and 3 show that the sum and product of two polynomial splines remain polynomial splines. Moreover, we can determine its B-spline coefficients from the B-spline coefficients of its constituents. Properties 1–3 provide a powerful tool to impose semi-infinite constraints on a polynomial function of splines by means of its B-spline coefficients.

Indeed, by parameterizing each of the components of the flat output, y_i as a polynomial spline, we can approximate the solution of the semi-infinite problem (3). Let Y collect the B-spline of the flat output, where the ith row, $\mathsf{y}_{i,:}$, corresponds to the B-spline coefficients of y_i. Since $\boldsymbol{\psi}_x$, $\boldsymbol{\psi}_u$ and $\mathbf{h}$ are assumed to be polynomial we can determine the B-spline coefficients $\mathsf{h}_i(\mathsf{Y})$ of the components of $\mathbf{h}(\boldsymbol{\psi}_x(\mathbf{y}, \mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(r-1)}), \boldsymbol{\psi}_u(\mathbf{y}, \mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(r)}))$. Then, an approximate solution for (3) is determined by solving

$$\begin{aligned} \underset{\mathsf{Y}}{\text{minimize}} \quad & \tilde{g}(\mathsf{Y}, t_\mathrm{f}) \\ \text{subject to} \quad & \mathsf{y}_{:,1}^{(i)} = \mathbf{y}_{0,i}, \qquad i = 0, \ldots, r-1 \\ & \mathsf{y}_{:,\mathrm{end}}^{(i)} = \mathbf{y}_{t_\mathrm{f},i}, \quad i = 0, \ldots, r-1 \\ & \mathsf{h}_i(\mathsf{Y}) \geq 0. \end{aligned} \tag{4}$$

Here, $\tilde{g}$ denotes the result of the substitution of the spline parameterization in the objective function of (3) and $\mathsf{y}_{j,:}^{(i)}$ denotes the B-spline coefficients of the ith derivative of y_j, which depend linearly on $\mathsf{y}_{j,:}$ [11].

Note that the boundary conditions of the problem can (and should) be directly eliminated, which improves numerical performance and conditioning. For linear systems, convex $\tilde{g}$ and concave $\mathbf{h}$ problem (4) is a convex optimization problem and can hence efficiently be solved to its global optimum. Even though for nonlinear system the problem is generally nonconvex, often (local) solutions can still be found efficiently due to the small number of optimization variables and the good numerical conditioning of the problem.

B. Reducing conservatism

Imposing a semi-infinite constraint on a polynomial spline through constraints on its B-spline coefficients yields only sufficient conditions and hence the optimal value of (4) yields an upper bound on the optimal value of (3). This conservatism is due to the distance between the control polygon of the spline and the spline itself and is essentially determined by two factors: the knots and the order. Hence, inserting knots or elevating the spline order without changing the spline's shape are two strategies that reduce the conservatism.

More precisely, let $s \in \Pi_{k,\boldsymbol{\kappa},\boldsymbol{\nu}}$ with B-spline coefficients s. Let $\Pi_{k,\boldsymbol{\kappa},\boldsymbol{\nu}} \subset \Pi_{\hat{k},\hat{\boldsymbol{\kappa}},\hat{\boldsymbol{\nu}}}$ with $\hat{k} \geq k$, and $\hat{\boldsymbol{\kappa}}$ and $\hat{\boldsymbol{\nu}}$ the refined break and continuity vectors such that $\boldsymbol{\kappa} \subseteq \hat{\boldsymbol{\kappa}}$ and $\nu_i \geq \hat{\nu}_j, i = 1, \ldots, m$ for some j. Then $\hat{s} \in \Pi_{\hat{k},\hat{\boldsymbol{\kappa}},\hat{\boldsymbol{\nu}}}$ with B-spline coefficients

$$\hat{\mathsf{s}} = T_{\hat{s}}^s \mathsf{s}$$

[1]Note that the multiplicity of a given knot could be even lower.

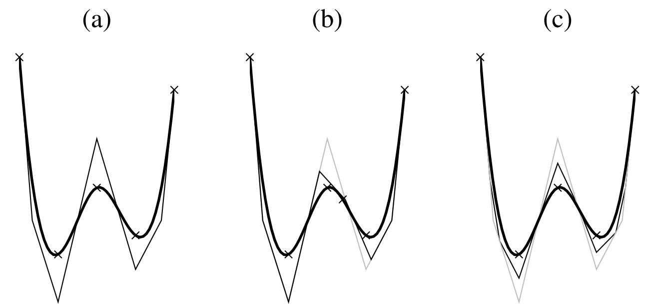

Fig. 2. Refining the control polygon brings the control polygon closer to the spline. (a) the original spline, (b) one knot insertion, (c) elevate order by one. Note that knot insertion acts locally, while order elevation changes the control polygon globally.

equals s and the control polygon of $\hat{s}$ will lie closer to the function than that of s.

A refinement of the break and/or continuity vectors acts locally on the constraint and can target specific regions where conservatism is high. Order elevation is a global approach and changes the entire shape of the control polygon. This is illustrated in Fig. 2. In both cases it is clear that the new control polygon gives a better representation of the spline than the original one.

V. Numerical examples

A. A linear benchmark system

This example considers the motorized base-stage high-precision positioning system, described in [7] and subsequently treated in [3] and [5]. With the system as depicted in Fig. 3, the transfer function matrices are given by

$$\begin{pmatrix} 25s^2 & 0 \\ 0 & 450s^2 + 1.1875 \times 10^4 s + 6.3955 \times 10^5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} u,$$

where the input u is the force applied to the stage, x_1 is the relative position of the center of mass of the stage with respect to a coordinate frame attached to the base with origin x_2, the position of the center of mass of the base in a fixed coordinate frame related to the ground [7]. A flat output y of the system is given by $y = x_1 - 0.0186\dot{x}_1 + 9.18x_2 - 0.3342\dot{x}_2$, such that

$$\begin{aligned} x_1 &= y + 0.0186\dot{y} + 7.0362 \times 10^{-4}\ddot{y} \\ x_2 &= -3.909 \times 10^{-5}\ddot{y} \\ u &= 25\ddot{y} + 0.4642\,\dddot{y} + 0.0176\ddddot{y}. \end{aligned} \tag{5}$$

The goal is to track a given reference x_1^{ref} for x_1, that interpolates between the system's boundary conditions, as accurately as possible, while restricting the movement of the base between −0.17 mm and 0.12 mm [5]. The system's initial state is given by $x_1(0) = 0\,\mathrm{m}$, $\dot{x}_1(0) = 0\,\mathrm{m\,s^{-1}}$, $x_2(0) = 0\,\mathrm{m}$, $\dot{x}_2(0) = 0\,\mathrm{m\,s^{-1}}$, $u(0) = 0\,\mathrm{N}$ and its terminal state by $x_1(0.2) = 0.02\,\mathrm{m}$, $\dot{x}_1(0.2) = 0\,\mathrm{m\,s^{-1}}$, $x_2(0.2) = 0\,\mathrm{m}$, $\dot{x}_2(0.2) = 0\,\mathrm{m\,s^{-1}}$, $u(0.2) = 0\,\mathrm{N}$. Thus the boundary conditions for the flat output are $y(0) = 0$, $y(0.2) = 0.02$ and zero for the derivatives up to order four. Now, x_1^{ref} is determined using (5) using an interpolating polynomial for the flat output. Then, the semi-infinite optimization problem is

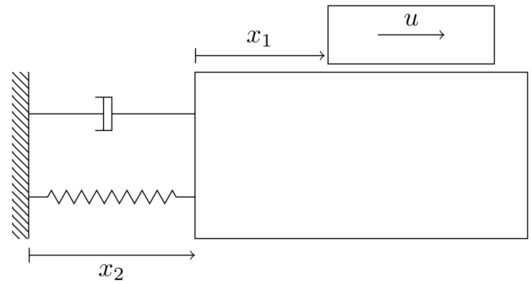

Fig. 3. Base stage high-precision positioning system

$$\begin{aligned} \underset{y(\cdot)}{\text{minimize}} \quad & \int_0^{0.2} \|x_1(y(t)) - x_1^{\text{ref}}(t)\|_2^2 dt \\ \text{subject to} \quad & y(0) = 0,\ y(0.2) = 0.02 \\ & y^{(j)}(0) = 0, y^{(j)}(0.2) = 0,\ j = 1, \ldots, 4 \\ & -17 \times 10^{-4} \le x_2(y(t)) \le 12 \times 10^{-4}, \\ & \forall t \in [0, 0.2]. \end{aligned} \tag{6}$$

We apply the methodology from Section IV for a B-spline parameterization of order 10 of the flat output, resulting in a quadratic program. For the optimization problem to be feasible minimally six equidistant internal breaks are required, yielding an optimal value $1.6081 \times 10^{-5}\,\mathrm{m^2 s}$. The optimization problem has 6 variables and 12 constraints. The resulting movement for the base and stage are illustrated in Fig. 4 in black. Clearly, as the true bounds do not become active, the solution is quite conservative. By applying the methodology from Section IV-B, we can reduce this conservatism considerably. The dark gray and light gray lines in Fig. 4 illustrate the base and stage movements for one and two midpoint refinements of the break vector of y. The corresponding optimal values are $8.4015 \times 10^{-6}\,\mathrm{m^2 s}$ and $6.3977 \times 10^{-6}\,\mathrm{m^2 s}$ respectively. The optimization problems have the same number of variables, yet the number of constraint increase to 26 and 54 respectively. For comparison the similar framework from [5] is also implemented. For this method minimally sixteen equidistant internal knots are required for the problem to be feasible. The optimal value is $3.1933 \times 10^{-5}\,\mathrm{m^2 s}$, which is higher than our method using only six internal knots. Moreover the computational time is larger, due to a greater number of variables and constraints in the problem formulation.

B. Quadrotor

This section considers the optimal control of a first principles quadrotor model in the vertical plane as shown in Fig. 5 to illustrate applicability on nonlinear systems. The quadrotor is controlled by two inputs, the total thrust force $u_1 = F_T/m$

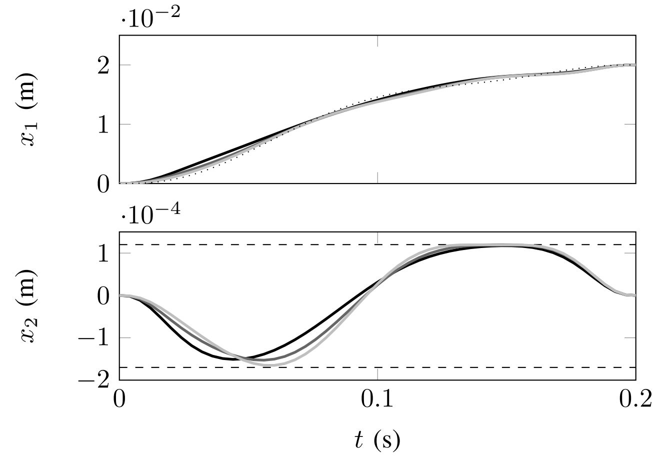

Fig. 4. Optimal tracking of the dotted reference for a base-stage high-precision positioning system, while limiting the displacement of the base. The black, gray and light gray lines show the solution for a spline with six internal knots using linear relaxations with respectively no, one and two midpoint refinements.

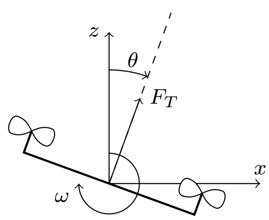

Fig. 5. First principles quadrotor model in the vertical plane and its controls

(to make the results independent of mass) and the pitch rate $u_2 = \omega$, and has three degrees of freedom, the horizontal and vertical position x and z, and the pitch angle θ.

The equations of motion are

$$\ddot{x} = \frac{F_T}{m}\sin\theta, \tag{7}$$

$$\ddot{z} = \frac{F_T}{m}\cos\theta - g, \tag{8}$$

$$\dot{\theta} = \omega, \tag{9}$$

where g denotes the gravitational acceleration and m is the quadrotor's mass. This system is differentially flat with flat output $\mathbf{y} = (x, z)$. Indeed, from the first two equations it follows that the state

$$\theta = \arctan\frac{\ddot{y}_1}{\ddot{y}_2 + g}.$$

We also find the input as a function of the flat output

$$u_1 = \sqrt{(\ddot{y}_2 + g)^2 + \ddot{y}_1^2},$$
$$u_2 = \frac{\dddot{y}_1(\ddot{y}_2 + g) - \ddot{y}_1\dddot{y}_2}{(\ddot{y}_2 + g)^2 + \ddot{y}_1^2}.$$

We now consider the problem of minimizing the thrust

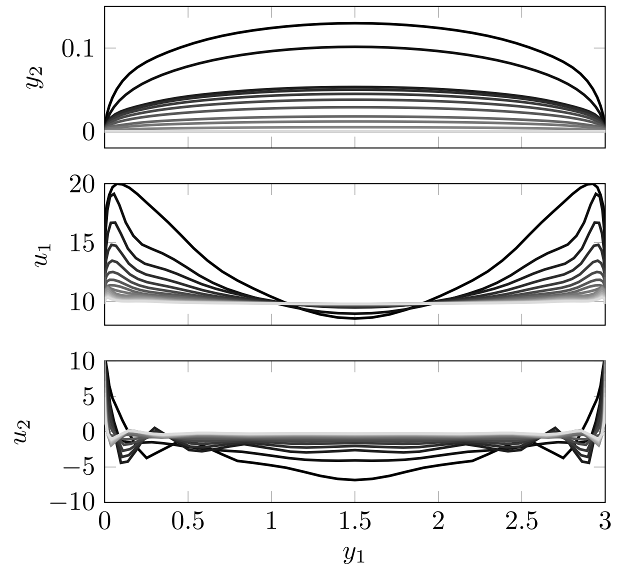

Fig. 6. Minimal thrust trajectories (top), thrust forces (middle) and rotational speeds (bottom) for varying final times

force of the quadrotor for a horizontal displacement of 3 m.

$$\begin{aligned}
\underset{\mathbf{y}(\cdot)}{\text{minimize}} \quad & \int_0^{t_\mathrm{f}} u_1(\mathbf{y}(t))^2 dt \\
\text{subject to} \quad & y_1(0) = 0,\, y_1(t_\mathrm{f}) = 3 \\
& y_2(0) = 0,\, y_2(t_\mathrm{f}) = 0 \\
& y_i^{(j)}(0) = 0, y_i^{(j)}(t_\mathrm{f}) = 0,\, i = 1, 2,\, j = 1, 2, 3 \\
& 1 \le u_1(\mathbf{y}(t)) \le 20,\, \forall t \in [0, 1] \\
& -10 \le u_2(\mathbf{y}(t)) \le 10,\, \forall t \in [0, 1].
\end{aligned} \tag{10}$$

Note that neither u_1 nor u_2 is polynomial in the flat output. However, squaring the constraint on u_1 yields

$$1 \le (\ddot{y}_2 + g)^2 + \ddot{y}_1^2 \le 400,$$

and multiplying the constraint on u_2 by its (nonnegative) denominator yields

$$\begin{aligned}
-10\left((\ddot{y}_2 + g)^2 + \ddot{y}_1^2\right) \le \dddot{y}_1(\ddot{y}_2 + g) - \ddot{y}_1\dddot{y}_2 \\
\le 10\left((\ddot{y}_2 + g)^2 + \ddot{y}_1^2\right),
\end{aligned}$$

resulting in polynomial constraints.

Having transformed the constraints, the optimal solution is calculated for 16 different final times, t_f, between 1 s and 2.5 s. The resulting trajectories and controls are illustrated in Fig. 6. The line color indicates the duration of the trajectory, black being the shortest and gray the longest. For longer trajectories the average thrust reduces, yet the value of the objective function increases after 1.5 s as shown in Fig. 7. This is due to a required minimal thrust force to keep the quadrotor airborne.

VI. Conclusion

This paper presents a novel way for dealing with semi-infinite constraints that arise in optimal control problems for

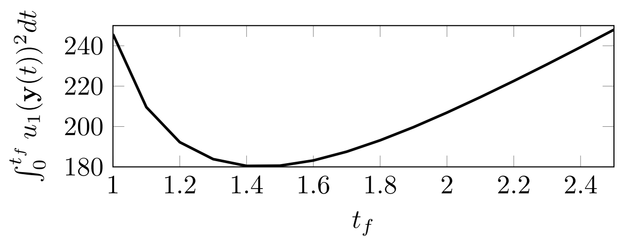

Fig. 7. Value of the objective function for varying final times. It features a clear minimum around 1.8 s

differentially flat systems that admit a polynomial representation. By means of a polynomial spline parameterization, semi-infinite constraints on states and inputs are imposed by constraining their B-spline coefficients instead, resulting in a small dimensional optimization problem that can be solved efficiently with modern-day solvers. The conservatism arising from these only sufficient constraints is reduced by means of knot insertions and/or degree elevations.

Future research will focus on extending the approach to general differentially flat systems by approximating the nonlinear constraints by polynomial constraints. This way a broader class of systems can be targeted, such as robotic manipulators, chemical reactors, ... Moreover, due to the small dimensions of the problem and the constant improvements in computational power, solution times may be sufficiently short to be performed on line, such that we may perform these tasks in a full horizon model predictive control setup.

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